Reliability Confidence Limits with the Weibull Distribution

The Weibull probability density function is

$$f(x) = \frac{\beta x^{\beta-1}}{\theta^{\beta}} \exp\left(\frac{x}{\theta}\right)^{\beta}, \; x \geq 0 \tag{1}$$

where: β is the shape parameter, and

θ is the scale parameter.

In some cases, a three-parameter Weibull distribution provides a better fit than the two-parameter Weibull distribution. The difference in the two distributions is the location parameter, δ, which shifts the distribution along the *x*-axis. By definition, there is a zero probability of failure for $x < \delta$. Although unusual, the location can be negative; this implies that items were failed prior to testing. The three parameter Weibull distribution is

$$f(x) = \frac{\beta (x-\delta)^{\beta-1}}{\theta^{\beta}} \exp\left(\frac{x-\delta}{\theta}\right)^{\beta}, \; x \geq \delta \qquad .2)$$

Maximum likelihood estimation

The maximum likelihood equations for the Weibull distribution are

$$\frac{1}{r}\sum_{i=1}^{r} \ln(x_i) = \left[\sum_{i=1}^{n} x_i^{\beta} \ln(x_i)\right]\left[\sum_{i=1}^{n} x_i^{\beta}\right]^{-1} - \frac{1}{\beta} \tag{3}$$

$$\hat{\theta} = \left[\frac{1}{r}\sum_{i=1}^{n} x_i^{\hat{\beta}}\right]^{1/\hat{\beta}} \tag{4}$$

where: *r* is the number of failures, and

n is the total number of data points, both censored and uncensored.

Iterative techniques are required to solve Equation 3. The estimated parameters are asymptotically normal. The variances of the estimates can be found by inverting the local information matrix. The local information matrix is

$$F = \begin{bmatrix} -\frac{\partial^2 L}{\partial \beta^2} & -\frac{\partial^2 L}{\partial \beta \partial \theta} \\ -\frac{\partial^2 L}{\partial \beta \partial \theta} & -\frac{\partial^2 L}{\partial \theta^2} \end{bmatrix} \tag{5}$$

The second partial derivatives of the log-likelihood equation are

$$\frac{\partial^2 L}{\partial \beta^2} = \sum_r \left[-\frac{1}{\beta} - \left(\frac{x_i}{\theta}\right)^\beta \ln^2\left(\frac{x_i}{\theta}\right) \right] + \sum_k \left[-\left(\frac{x_i}{\theta}\right)^\beta \ln^2\left(\frac{x_i}{\theta}\right) \right] \tag{6}$$

$$\frac{\partial^2 L}{\partial \theta^2} = \sum_r \left[\frac{\beta}{\theta^2} - \left(\frac{x_i}{\theta}\right)^\beta \left(\frac{\beta}{\theta^2}\right)(\beta+1) \right] + \sum_k \left[-\left(\frac{x_i}{\theta}\right)^\beta \left(\frac{\beta}{\theta^2}\right)(\beta+1) \right] \tag{7}$$

$$\begin{aligned} \frac{\partial^2 L}{\partial \beta \partial \theta} &= \sum_r \left\{ -\frac{1}{\theta} + \left(\frac{x_i}{\theta}\right)^\beta \left(\frac{1}{\theta}\right)\left[\beta \ln\left(\frac{x_i}{\theta}\right) + 1\right] \right\} \\ &+ \sum_k \left\{ \left(\frac{x_i}{\theta}\right)^\beta \left(\frac{1}{\theta}\right)\left[\beta \ln\left(\frac{x_i}{\theta}\right) + 1\right] \right\} \end{aligned} \tag{8}$$

where: $\sum_r$ represents summation over all failures, and

$\sum_k$ represents summation over all censored points.

The variances of the estimated parameters are

$$F^{-1} = \begin{bmatrix} \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\beta},\hat{\theta}) \\ \mathrm{cov}(\hat{\beta},\hat{\theta}) & \mathrm{var}(\hat{\theta}) \end{bmatrix} \tag{9}$$

Approximate (1–α)100% confidence intervals for the estimated parameters are

$$\frac{\hat{\beta}}{\exp\left(\frac{K_{\alpha/2}\sqrt{\mathrm{var}(\hat{\beta})}}{\hat{\beta}}\right)} \le \beta \le \hat{\mu} + \hat{\beta}\exp\left(\frac{K_{\alpha/2}\sqrt{\mathrm{var}(\hat{\beta})}}{\hat{\beta}}\right) \tag{10}$$

$$\frac{\hat{\theta}}{\exp\left[\frac{K_{\alpha/2}\sqrt{\mathrm{var}(\hat{\theta})}}{\hat{\theta}}\right]} \le \theta \le \hat{\theta}\exp\left[\frac{K_{\alpha/2}\sqrt{\mathrm{var}(\hat{\theta})}}{\hat{\theta}}\right] \tag{11}$$

where $K_{\alpha/2}$ is the inverse of the standard normal probability density function.

These confidence intervals are approximate, but approach exactness as the sample size increases.

Confidence intervals for reliability can be found using the expressions

$$\begin{aligned} &\exp\left[-\exp\left(u + K_{a/2}\sqrt{\mathrm{var}(u)}\right)\right] \\ &\quad \le R(x) \le \exp\left[-\exp\left(u - K_{a/2}\sqrt{\mathrm{var}(u)}\right)\right] \end{aligned} \tag{12}$$

$$u = \beta[\ln(x) - \ln(\theta)] \tag{13}$$

$$\operatorname{var}(\hat{u}) \approx \beta^2 \left[\left(\frac{\operatorname{var}(\hat{\theta})}{\hat{\theta}^2} \right) + \left(\frac{\hat{u}^2 \operatorname{var}(\hat{\beta})}{\beta^4} \right) - \left(\frac{2\hat{u} \operatorname{cov}(\hat{\beta}, \hat{\theta})}{\hat{\beta}^2 \hat{\theta}} \right) \right] \tag{14}$$

Confidence intervals for percentiles can be found using the expressions

$$e^{y_L} \leq \hat{x} \leq e^{y_U} \tag{15}$$

$$\hat{x} = \hat{\theta} \left[-\ln(1-p) \right]^{1/\hat{\beta}} \tag{16}$$

$$y_L = \ln \hat{\theta} + \frac{\ln\left[-\ln(1-p) \right]}{\hat{\beta}} - K_\alpha \sqrt{\operatorname{var}(\hat{y})} \tag{17}$$

$$y_U = \ln \hat{\theta} + \frac{\ln\left[-\ln(1-p) \right]}{\hat{\beta}} + K_\alpha \sqrt{\operatorname{var}(\hat{y})} \tag{18}$$

$$\operatorname{var}(\hat{y}) = \frac{\operatorname{var}(\hat{\theta})}{\hat{\theta}^2} + \frac{\left\{ \ln\left[-\ln(1-p) \right] \right\}^2 \operatorname{var}(\hat{\beta})}{\beta^4} - \frac{2\left\{ \ln\left[-\ln(1-p) \right] \right\} \operatorname{cov}(\hat{\theta}, \hat{\beta})}{\hat{\beta}^2 \hat{\theta}} \tag{19}$$