The exponential distribution

The exponential distribution is a one-parameter, continuous distribution. It is commonly expressed in terms of its mean, θ, and the inverse of its mean, λ. The exponential probability density function is

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} = \lambda e^{-\lambda x}, \quad x \geq 0 \qquad (1)$$

where: θ = the distribution mean, and
$\lambda = 1/\theta$ = the failure rate.

The exponential probability density function is shown in Figure 1.

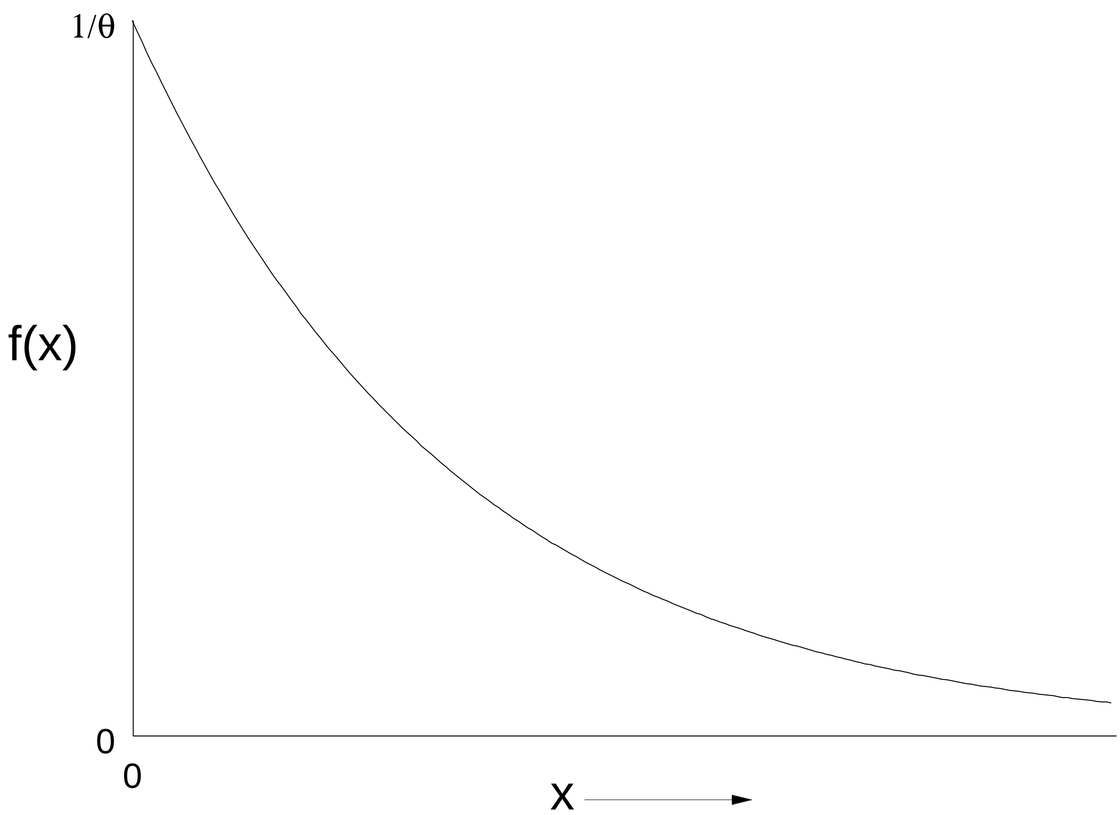

Figure 1. The exponential probability density function.

The exponential reliability function is

$$R(x) = e^{-\frac{x}{\theta}} = e^{-\lambda x}\ ,\ x \geq 0 \tag{2}$$

This is shown in Figure 2.

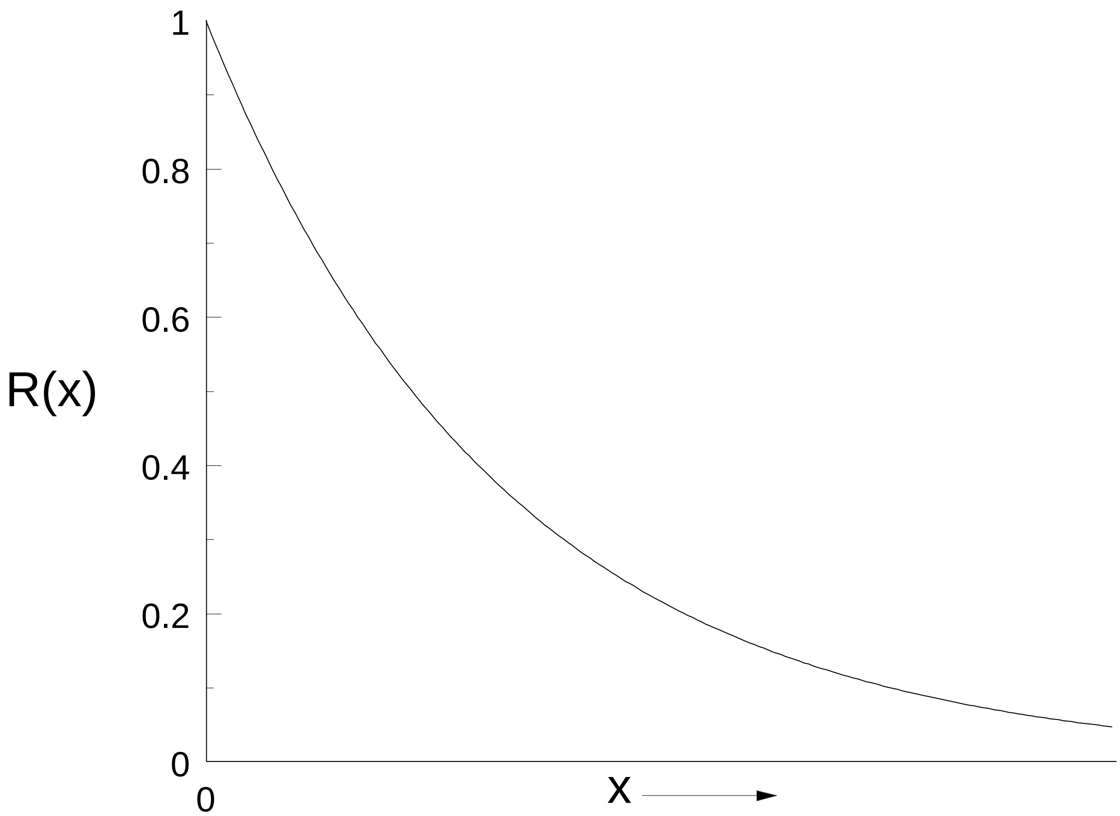

Figure 2. The exponential reliability function.

The exponential hazard function is

$$h(x) = \frac{f(x)}{R(x)} = \frac{1}{\theta} = \lambda \tag{3}$$

The exponential hazard function is constant, as shown in Figure 3. A constant failure rate is unique to the exponential distribution, and is responsible for the "lack of memory" property. "Lack of memory" means the probability of failure in a specific time interval is the same regardless of the starting point of that time interval.

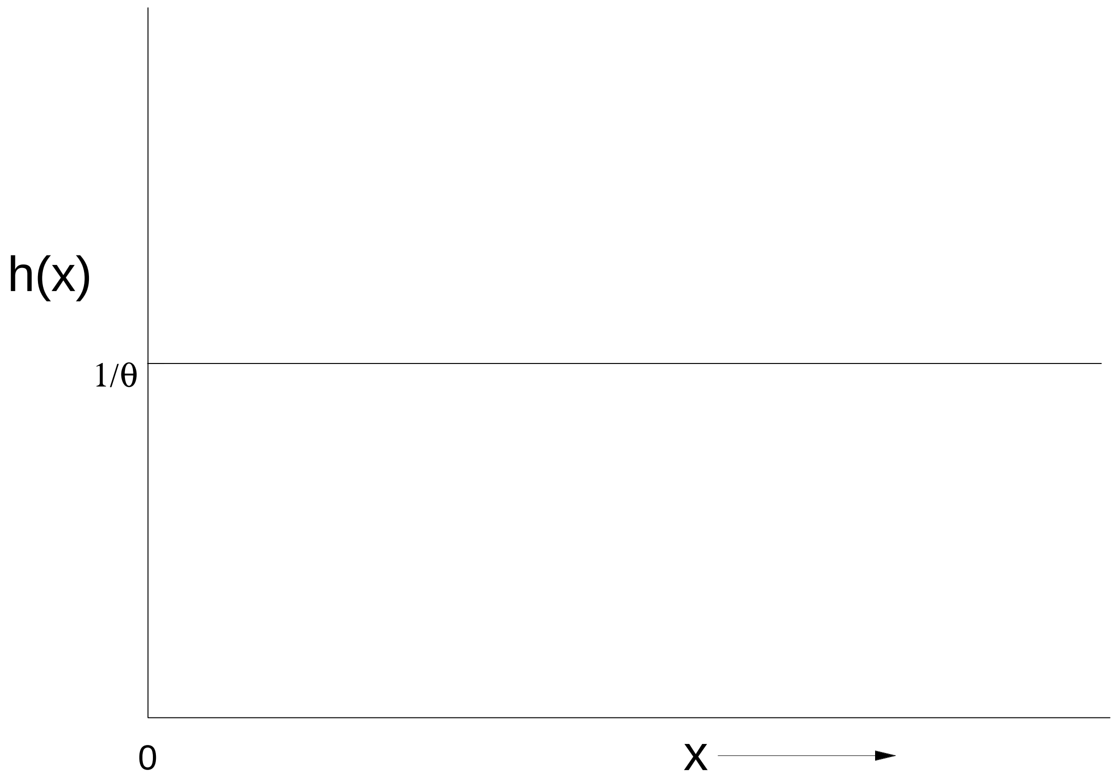

Figure 3. The exponential hazard function.

For example, if an item follows the exponential distribution, the probability of failure in the interval $x = 0$ to $x = 20$ is the same as the probability of failure in the interval $x = 100$ to $x = 120$.

The variance of the exponential distribution is

$$V(x) = \theta^2 = \frac{1}{\lambda^2} \tag{4}$$

Example 3.1

What is the probability of an item surviving until t = 100 units if the item is exponentially distributed with a mean time between failure of 80 units? Given that the item survived to 200 units, what is the probability of survival until t = 300 units? What is the value of the hazard function at 200 units, 300 units?

Solution

The probability of survival until t = 100 units is

$$R(100) = e^{-\left(\frac{100}{80}\right)} = 0.2865$$

The probability of survival until t = 300 units given survival until t = 200 units is

$$R(300,200) = \frac{R(300)}{R(200)} = \frac{e^{-300/80}}{e^{-200/80}} = 0.2865$$

Note that this is equal to the probability of failure in the interval from t=0 to t=100.

The value of the hazard function is equal to the failure rate and is constant $h(t)$= 1/80 = 0.125